PLEASE ANSWER ALL QUESTIONS.
PLEASE EXPLAIN YOUR ANSWERS.

1. Consider the following game G.

		Player 2 L	C	R
Player 1	U	6, 3	3, 0	0, 1
	M	5, 4	1, 7	4, 3
	D	2, 5	5, 7	0, 0

(a) Solve the game by iterated elimination of strictly dominated strategies. If you get a unique solution, indicate this. If your solution is not unique, write up the reduced game where you have eliminated the strictly dominated strategies.

Solution: For player 2, R is strictly dominated by L. After eliminating R, M is strictly dominated by U for player 1. There are no more strictly dominated strategies. This leaves us with the following game.

		Player 2 L	C
Player 1	U	6, 3	3, 0
	D	2, 5	5, 7

(b) Find all the pure and mixed-strategy Nash Equilibria.[1]

Solution: Since a strictly dominated strategy will never be played in equilibrium, we can just solve the 2x2 game obtained in the previous question.

		Player 2 L	C
Player 1	U	**6, 3**	3, 0
	D	2, 5	**5, 7**

The pure-strategy NE are $\{(U, L), (D, C)\}$.

Suppose player 1 plays U with probability p and player 2 plays L with probability q. Then player 1 is indifferent between playing U and D whenever

$$q(6) + (1 - q)(3) = q(2) + (1 - q)(5).$$

This solves for $q = 1/3$. Conversely, player 2 is indifferent between playing L and C whenever

$$p(3) + (1 - p)(5) = p(0) + (1 - p)(7).$$

This solves for $p = 2/5$. Thus, the mixed-strategy NE using the notation in the footnote is $(p_1^*, p_2^*, p_3^*; q_1^*, q_2^*, q_3^*) = (\frac{2}{5}, 0, \frac{3}{5}; \frac{1}{3}, \frac{2}{3}, 0)$.

(c) It has been argued that randomization in decision making lacks 'behavioral support'. Give one (and just one) example of a different interpretation of mixed strategies, that does not rely on players actually randomizing.

Solution: The interpretation could for instance be that of a large population of players, all of whom play pure strategies. However, if there is random matching and players cannot observe the actual strategy of the other player, this is (in our simple setting) as if they were playing against a player with a mixed strategy. Other interpretations are also accepted.

[1] For the mixed-strategy equilibria, you can assume that player 1 plays U with probability p_1, M with probability p_2 and D with probability $1 - p_1 - p_2$. Similarly, assume that player 2 plays L with probability q_1, C with probability q_2 and R with probability $1 - q_1 - q_2$.

(d) Suppose we repeat the game twice. Let the new game be denoted $G(2)$. Find a Subgame-perfect Nash Equilibrium of $G(2)$ and write it up formally. (Any equilibrium will do, you do NOT have to find all equilibria.)

Solution: Any SPNE will do. For instance, playing (U, L) in all subgames. This is a SPNE since a NE is played in every subgame. It is important to mention this in the analysis.

2. Two tech entrepreneurs have made 1 dollar from selling a new app and need to decide how to allocate the gains. If no agreement is reached, neither entrepreneur gets anything. Let x_1 and x_2 be the amounts that entrepreneur 1 and 2 get in the negotiation. Their utilities are:

$$u_1(x_1) = 4x_1$$
$$u_2(x_2) = 2\sqrt{x_2}.$$

(a) Can the axioms Pareto efficiency (PAR), Symmetry (SYM) and Invariance to equivalent payoff representations (INV) be used to conclude that the Nash Bargaining Solution must satisfy $v_1^* = v_2^*$? Explain briefly.

Solution: No. In this example, the payoff representations of the two players are not equivalent, since player 2's payoff function is not an affine transformation of player 1's payoff function. Therefore, the first three axioms are not sufficient to solve the problem.

(b) Find the Nash Bargaining Solution. What are the allocations?

Solution: The disagreement allocation is $D = (0, 0)$, which corresponds to $d = (0, 0)$. We must solve the program $\max_{v_1,v_2}(v_1 - d_1)(v_2 - d_2)$ subject to $(v_1, v_2) \in U$. Doing the transformations, we get $x_1 = \frac{v_1}{4}$ and $x_2 = \frac{v_2^2}{4}$. The solution must be efficient, so we can substitute $\frac{v_1}{4} + \frac{v_2^2}{4} = 1$, i.e. $v_1 + v_2^2 = 4$, into the problem, along with $d_1 = d_2 = 0$. Thus: $(v_1 - d_1)(v_2 - d_2) = v_1 v_2 = (4 - v_2^2)v_2$. Take the first-order condition: $4 - 3v_2^2 = 0$. This gives $v_2^* = \frac{2}{\sqrt{3}}$. Then $v_1^* = 4 - (\frac{2}{\sqrt{3}})^2 = \frac{8}{3}$. This corresponds to the allocations $x_1^* = \frac{8}{12}$ and $x_2^* = \frac{4}{12}$.

(c) Now, suppose the entrepreneurs have signed a contract before they started the venture, guaranteeing that in case of disagreement, entrepreneur 1 gets to keep 0.5 dollar whereas entrepreneur 2 gets nothing. Find the Nash Bargaining Solution. What are the allocations?

Solution: The disagreement allocation is $D = (\frac{1}{2}, 0)$, which corresponds to $d = (2, 0)$. We must solve the program $\max_{v_1,v_2}(v_1 - d_1)(v_2 - d_2)$ subject to $(v_1, v_2) \in U$. Doing the transformations, we get $x_1 = \frac{v_1}{4}$ and $x_2 = \frac{v_2^2}{4}$. The solution must be efficient, so we can substitute $\frac{v_1}{4} + \frac{v_2^2}{4} = 1$, i.e. $v_1 + v_2^2 = 4$ into the problem, along with $d_1 = 0.5$ and $d_2 = 0$. Thus: $(v_1 - d_1)(v_2 - d_2) = (v_1 - 2)v_2 = (4 - v_2^2 - 2)v_2 = (2 - v_2^2)v_2$. Take the first-order condition: $2 - 3v_2^2 = 0$. This gives $v_2^* = \sqrt{\frac{2}{3}}$. Then $v_1^* = 4 - (\sqrt{\frac{2}{3}})^2 = \frac{10}{3}$. This corresponds to the allocations $x_1^* = \frac{10}{12}$ and $x_2^* = \frac{2}{12}$.

(d) Compare the answers in (b) and (c). If the allocations are the same, explain why this is the case. If they are different, explain why this is the case.

Solution: Player 1 gets a better allocation in the second problem, because his disagreement point has improved. Intuitively speaking, the Nash solution gives each player their disagreement payoffs plus an equitable split of the surplus from bargaining.

3. Consider the entry game represented in Figure 1, in which the incumbent can be *weak* ($i = w$) or *strong* ($i = s$). Here, the incumbent **does not** know his own type, but the outsider does.[2] You can think of this as a probability θ that the outsider has found the incumbent's 'weak spot'. Suppose $\theta \in (0, 1)$.

   The timing of the game is as follows. The outsider must first decide whether to enter (E_i) or not (N_i). (Here the i indicates the type of the incumbent, since the outsider conditions his choice on the incumbent's type.) If he doesn't enter, we assume that the game ends. If he enters, on the other hand, the incumbent can choose either to fight (F) or acquiesce (A). If he acquiesces, the game ends. If he fights, the game continues. In this case, the outsider must decide whether to stay (S_i) or leave (L_i). (Again, $i = w, s$.)

   Suppose the incumbent's beliefs in his information set attach probability p to him being the weak type. The payoffs are as indicated in Figure 1. *The first payoff is that of the incumbent, the second is that of the outsider.*

   (a) Indicate how many strategies each player has, and write up one such strategy for each player. Is this a game of imperfect or incomplete information?

   Solution: The incumbent has two strategies, e.g. F. The outsider has $2^4 = 16$ strategies, e.g. (E_w, N_s, S_w, L_s). The game is of incomplete information.

   (b) Show that for certain values of θ, there is an equilibrium in which the outsider always enters (plays E_i for $i = w, s$) and the incumbent acquiesces (plays A). Be careful to specify how the equilibrium depends on p and θ. (Hint: Use Bayes' Rule to calculate p given θ and given that the outsider always enters.)

   Solution: Starting from the end of the game, it is always optimal for the outsider to play S_w and L_s. Therefore, given beliefs p, the incumbent's expected payoff from playing F is $p(0) + (1 - p)(3) = 3(1 - p)$ whereas his expected payoff from playing A is $p(2) + (1 - p)(1) = 1 + p$. Thus, it is optimal to play F whenever $3(1 - p) \geq 1 + p \Leftrightarrow p \leq \frac{1}{2}$. Thus, in any equilibrium the incumbent's strategy must be

$$s_I(p) = \begin{cases} F \text{ if } p \leq \dfrac{1}{2} \\ A \text{ if } p \geq \dfrac{1}{2}. \end{cases}$$

   Regardless of the type of the incumbent, the outsider will only enter if the incumbent plays A. Suppose the outsider always enters. Then Bayes' Rule implies $p = \theta$. Thus, an equilibrium of this type exists only if $\theta \geq \frac{1}{2}$.
   Thus, the equilibrium is $(A, E_w E_s S_w L_s; p = \theta)$ for $\theta \geq \frac{1}{2}$.

   (c) Show that there is also an equilibrium in which the outsider never enters (he plays N_i for $i = w, s$). Be careful when you write up the equilibrium to specify p, and how the equilibrium depends on θ. (Hint: In this case, Bayes' Rule does not apply to p.)

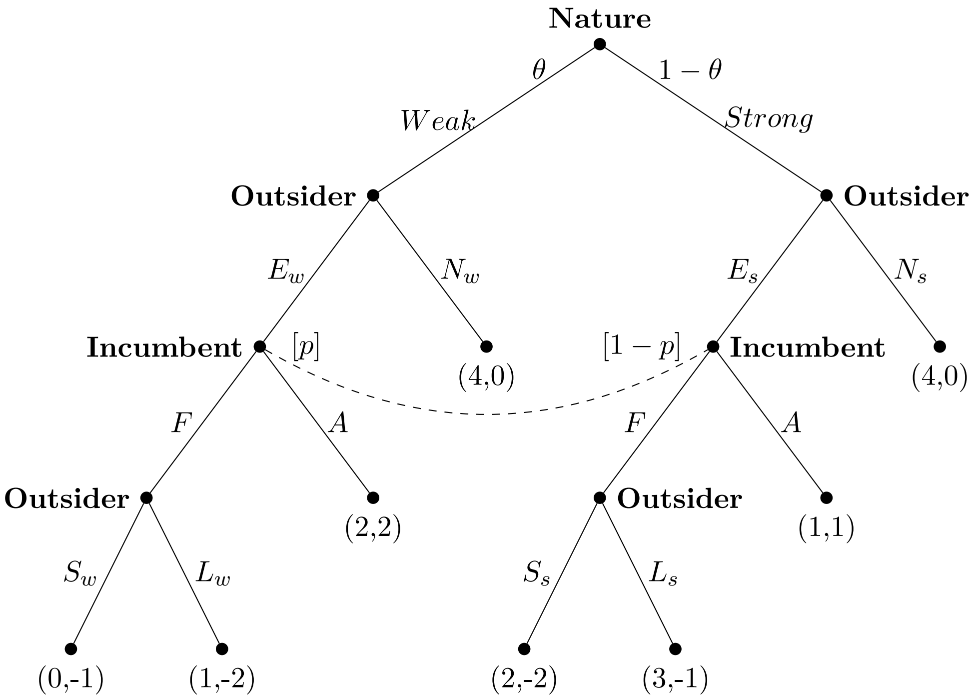
   Solution: Now, the incumbent's information set is off the equilibrium path. Therefore, p is unrestricted by Bayes' Rule. The outsider's strategy in his last information sets is the same as before. So is the strategy of the incumbent, conditional on p. In order for it to be optimal for the outsider never to enter, we need that the incumbent plays F. This will be optimal if $p \leq \frac{1}{2}$.
   Thus, the equilibrium is $(F, N_w N_s S_w L_s; p \leq \frac{1}{2})$ for all θ.

   (d) Consider the equilibrium in (c) where the outsider never enters. Does it satisfy SR5 ('strict domination')?

   Solution: Both outsider types can potentially do better from entering (if the incumbent acquiesces) and therefore we cannot apply SR5.

[2] Notice that this is the 'opposite' of the entry game you saw in the lectures.

Figure 1: Entry game



4. Consider a *first-price sealed bid auction* with two bidders, who have valuations v_1 and v_2, respectively. For $i = 1, 2$, these values are distributed independently uniformly with

$$v_i \sim u(1, 2).$$

Thus, the values are *private*.

(a) Suppose player j uses the strategy $b_j(v_j) = cv_j + d$. For $i \neq j$, show that conditional on this strategy, the probability that i wins when he bids b_i is

$$\mathbb{P}(i \text{ wins}|b_i) = \frac{b_i - d - c}{c},$$

whenever $c + d \leq b_i \leq 2c + d$.

Solution: Notice that $\mathbb{P}(b_i > b_j(v_j)) = \mathbb{P}\left(\frac{b_i-d}{c} > v_j\right) = \frac{\frac{b_i-d}{c}-1}{2-1} = \frac{b_i-d-c}{c}$ whenever $c + d \leq b_i \leq 2c + d$. Outside the bounds, the probability is 0 or 1, respectively.

(b) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$, $i = 1, 2$. Find c and d.

Solution. We follow the procedure seen in the lecture. Assume that bidder j follows his proposed equilibrium strategy $b_j(v_j) = cv_j + d$. Then calculate the expected payoff to i from bidding b_i:

$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins}|b_i)(v_i - b_i) \\
&= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i)
\end{aligned}$$

Thus

$$\mathbb{E}[u_i(b_i, v_i)] = \frac{b_i - d - c}{c}(v_i - b_i).$$

Take the first-order condition

$$\frac{1}{c}\left[(v_i - 2b_i) + (d + c)\right] = 0.$$

Easy to check SOC. Hence, best response is

$$b_i(v_i) = \frac{1}{2}v_i + \frac{1}{2}(d + c).$$

Therefore, $c^* = \frac{1}{2}$ and $d^* = \frac{1}{2}(d^* + c) = \frac{1}{2}(d^* + \frac{1}{2})$, which solves for $d^* = \frac{1}{2}$. I.e. $b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2}$.